

KEY POLYNOMIALS AND PSEUDO-CONVERGENT SEQUENCES

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ABSTRACT. In this paper we introduce a new concept of key polynomials for a given valuation ν on $K[x]$. We prove that such polynomials have many of the expected properties of key polynomials as those defined by MacLane and Vaquié, for instance, that they are irreducible and that the truncation of ν associated to each key polynomial is a valuation. Moreover, we prove that every valuation ν on $K[x]$ admits a sequence of key polynomials that completely determines ν (in the sense which we make precise in the paper). We also establish the relation between these key polynomials and pseudo-convergent sequences defined by Kaplansky.

1. INTRODUCTION

Given a valuation ν of a field K , it is important to understand what are the possible extensions of ν to $K[x]$. Many different theories have been developed in order to understand such extensions. For instance, in [3], MacLane develops the theory of key polynomials. He proves that given a discrete valuation ν of K , every extension of ν to $K[x]$ is uniquely determined by a sequence (with order type at most ω) of key polynomials. Recently, M. Vaquié developed a more general theory of key polynomials (see [9]), which extends the results of MacLane for a general valued field (that is, the given valuation of K is no longer assumed to be discrete). At the same time, F.H. Herrera Govantes, W. Mahboub, M.A. Olalla Acosta and M. Spivakovsky developed another definition of key polynomials (see [5]). This definition is an adaptation of the concept of generating sequences introduced by Spivakovsky in [7]. A comparison between this two definitions of key polynomials is presented in [4].

Roughly speaking, for a given valuation μ of $K[x]$, a MacLane – Vaquié key polynomial $\phi \in K[x]$ for μ is a polynomial that allows us to obtain a new valuation μ_1 of $K[x]$ with $\mu_1(\phi) = \gamma_1 > \mu(\phi)$ and $\mu(p) = \mu_1(p)$ for every $p \in K[x]$ with $\deg(p) < \deg(\phi)$ (in this case we denote μ_1 by $[\mu; \mu_1(\phi) = \gamma_1]$). Then, for any valuation ν of $K[x]$ one tries to obtain a sequence of valuations $\mu_0, \mu_1, \dots, \mu_n, \dots$ with μ_0 a monomial valuation and $\mu_{i+1} = [\mu_i; \mu_{i+1}(\phi_{i+1}) = \gamma_{i+1}]$ for a key polynomial ϕ_{i+1} for μ_i , such that

$$(1) \quad \nu = \lim \mu_i$$

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(in the sense that will be defined precisely below). This process does not work in general, that is, the equality (1) may not hold even after one constructs an infinite sequence $\{\mu_i\}$. This leads one to introduce the concept of “limit key polynomial”. It is known that valuations which admit limit key polynomials are more difficult to handle. For instance, it was proved by J.-C. San Saturnino (see Theorem 6.5 of [6]), that if a valuation ν is centered on a noetherian local domain and ν does not admit limit key polynomials (on any sub-extension $R' \subseteq R'[x] \subseteq R$ with $\dim R' = \dim R - 1$), then it has the local uniformization property (where we assume, inductively, that local uniformization holds for R').

In this paper, we introduce a new concept of key polynomials. Let K be a field and ν a valuation on $K[x]$. Let Γ denote the value group of K and Γ' the value group of $K[x]$. For a positive integer b , let $\partial_b := \frac{1}{b!} \frac{\partial^b}{\partial x^b}$ (this differential operator of order b is sometimes called **the b -th formal derivative**). For a polynomial $f \in K[x]$ let

$$\epsilon(f) = \max_{b \in \mathbb{N}} \left\{ \frac{\nu(f) - \nu(\partial_b f)}{b} \right\}.$$

A monic polynomial $Q \in K[x]$ is said to be a **key polynomial** (of level $\epsilon(Q)$) if for every $f \in K[x]$ if $\epsilon(f) \geq \epsilon(Q)$, then $\deg(f) \geq \deg(Q)$.

This new definition offers many advantages. For instance, it gives a criterion to determine, for a given valuation ν of $K[x]$, whether any given polynomial is a key polynomial for ν . This has a different meaning than in the approach of MacLane-Vaquié. In their approach, a key polynomial allows us to “extend the given valuation” and here a key polynomial allows us to “truncate the given valuation”. For instance, our definition of key polynomials treats the limit key polynomials on the same footing as the non-limit ones. Moreover, we present a characterization of key polynomial (Theorem 2.12) which allows us to determine whether a given key polynomial is a limit key polynomial. A more precise comparison between the concept of key polynomial introduced here and that of MacLane – Vaquié will be explored in a forthcoming paper by Decoup, Mahboub and Spivakovsky.

Given two polynomials $f, q \in K[x]$ with q monic, we call the **q -standard expansion of f** the expression

$$f(x) = f_0(x) + f_1(x)q(x) + \dots + f_n(x)q^n(x)$$

where for each i , $0 \leq i \leq n$, $f_i = 0$ or $\deg(f_i) < \deg(q)$. For a polynomial $q(x) \in K[x]$, the **q -truncation of ν** is defined as

$$\nu_q(f) := \min_{0 \leq i \leq n} \{\nu(f_i q^i)\}$$

where $f = f_0 + f_1 q + \dots + f_n q^n$ is the q -standard expansion of f . In Section 2, we present an example that shows that ν_q does need to be a valuation. We also prove (Theorem 2.6) that if Q is a key polynomial, then ν_Q is a valuation. A set Λ of key polynomials is said to be a **complete set of key polynomials for ν** if for every $f \in K[x]$, there exists $Q \in \Lambda$ such that $\nu_Q(f) = \nu(f)$. One of the main results of this paper is the following:

Theorem 1.1. *Every valuation ν on $K[x]$ admits a complete set of key polynomials.*

Another way of describing extensions of valuations from K to $K[x]$ is the theory of pseudo-convergent sequences developed by Kaplansky in [1]. He uses this theory to determine whether a maximal immediate extension of the valued field (K, ν) is unique (up to isomorphism). For a valued field (K, ν) , a **pseudo-convergent sequence** is a well-ordered subset $\{a_\rho\}_{\rho < \lambda}$ of K , without last element, such that

$$\nu(a_\sigma - a_\rho) < \nu(a_\tau - a_\sigma) \text{ for all } \rho < \sigma < \tau < \lambda.$$

For a given pseudo-convergent sequence $\{a_\rho\}_{\rho < \lambda}$ it is easy to show that either $\nu(a_\rho) < \nu(a_\sigma)$ for all $\rho < \sigma < \lambda$ or there is $\rho < \lambda$ such that $\nu(a_\sigma) = \nu(a_\rho)$ for every $\rho < \sigma < \lambda$. If we set $\gamma_\rho := \nu(a_{\rho+1} - a_\rho)$, then $\nu(a_\sigma - a_\rho) = \gamma_\rho$ for every $\rho < \sigma < \lambda$. Hence, the sequence $\{\gamma_\rho\}_{\rho < \lambda}$ is an increasing subset of Γ . An element $a \in K$ is said to be a **limit** of the pseudo-convergent sequence $\{a_\rho\}_{\rho < \lambda}$ if $\nu(a - a_\rho) = \gamma_\rho$ for every $\rho < \lambda$.

One can prove that for every polynomial $f(x) \in K[x]$, there exists $\rho_f < \lambda$ such that either

$$(2) \quad \nu(f(a_\sigma)) = \nu(f(a_{\rho_f})) \text{ for every } \rho_f \leq \sigma < \lambda,$$

or

$$(3) \quad \nu(f(a_\sigma)) > \nu(f(a_\rho)) \text{ for every } \rho_f \leq \rho < \sigma < \lambda.$$

If case (2) happens, we say that the value of f is fixed by $\{a_\rho\}_{\rho < \lambda}$ (or that $\{a_\rho\}_{\rho < \lambda}$ fixes the value of f). A pseudo-convergent sequence $\{a_\rho\}_{\rho < \lambda}$ is said to be of **transcendental type** if for every polynomial $f(x) \in K[x]$ the condition (2) holds. Otherwise, $\{a_\rho\}_{\rho < \lambda}$ is said to be of **algebraic type**, i.e., if there exists at least one polynomial for which condition (3) holds.

The concept of key polynomials appears in the approach to local uniformization by Spivakovsky. On the other hand, the concept of pseudo-convergent sequence plays an important role in the work of Knaf and Kuhlmann (see [2]) on the local uniformization problem. In this paper, we present a comparison between the concepts of key polynomials and pseudo-convergent sequences. More specifically, we prove the following:

Theorem 1.2. *Let ν be a valuation on $K[x]$ and let $\{a_\rho\}_{\rho < \lambda} \subset K$ be a pseudo-convergent sequence, without a limit in K , for which x is a limit. If $\{a_\rho\}_{\rho < \lambda}$ is of transcendental type, then $\Lambda := \{x - a_\rho \mid \rho < \lambda\}$ is a complete set of key polynomials for ν . On the other hand, if $\{a_\rho\}_{\rho < \lambda}$ is of algebraic type, then every polynomial $q(x)$ of minimal degree among the polynomials not fixed by $\{a_\rho\}_{\rho < \lambda}$ is a limit key polynomial for ν .*

2. KEY POLYNOMIALS

We will assume throughout this paper that K is a field, ν a valuation of $K[x]$, non-trivial on K with $\nu(x) \geq 0$. We begin by making some remarks.

Remark 2.1. (i): Every linear polynomial $x - a$ is a key polynomial (of level $\epsilon(x - a) = \nu(x - a)$).

(ii): Take a polynomial $f(x) \in K[x]$ of degree greater than one and assume that there exists $a \in K$ such that $\nu(\partial_b f(a)) = \nu(\partial_b f(x))$ for every $b \in \mathbb{N}$ (note that such an a always exists if the assumptions of Theorem 1.2 hold and the pseudo-convergent sequence is transcendental or is algebraic and $\deg(f) \leq \deg(q)$). Write

$$f(x) = f(a) + \sum_{i=1}^n \partial_i f(a)(x - a)^i$$

and take $h \in \{1, \dots, n\}$ such that

$$\nu(\partial_h f(x)) + h\nu(x - a) = \min_{1 \leq i \leq n} \{\nu(\partial_i f(x)) + i\nu(x - a)\}.$$

If $\nu(f(a)) < \nu(\partial_h f(x)) + h\nu(x - a)$, then $\nu(f(x)) = \nu(f(a))$ and hence

$$\frac{\nu(f(x)) - \nu(\partial_i f(x))}{i} < \nu(x - a)$$

for every i , $1 \leq i \leq n$. Consequently, $\epsilon(f) < \nu(x - a) = \epsilon(x - a)$ and hence f is not a key polynomial. On the other hand, if

$$\nu(\partial_h f(x)) + h\nu(x - a) \leq \nu(f(a)),$$

then

$$(4) \quad \nu(f(x)) \geq \nu(\partial_h f(x)) + h\nu(x - a)$$

and if the equality holds in (4), then

$$\epsilon(f) = \frac{\nu(f(x)) - \nu(\partial_h f(x))}{h} = \nu(x - a) = \epsilon(x - a)$$

and hence f is not a key polynomial. In other words, the only situation when f may be a key polynomial is when

$$f(x) > \min_{1 \leq i \leq n} \{f(a), \nu(\partial_i f(x)) + i\nu(x - a)\}.$$

Remark 2.2. We observe that if Q is a key polynomial of level $\epsilon := \epsilon(Q)$, then for every polynomial $f \in K[x]$ with $\deg(f) < \deg(Q)$ and every $b \in \mathbb{N}$ we have

$$(5) \quad \nu(\partial_b(f)) > \nu(f) - b\epsilon.$$

Indeed, from the definition of key polynomial we have that $\epsilon > \epsilon(f)$. Hence, for every $b \in \mathbb{N}$ we have

$$\frac{\nu(f) - \nu(\partial_b(f))}{b} \leq \epsilon(f) < \epsilon$$

and this implies (5).

Let

$$I(f) = \left\{ b \in \mathbb{N} \mid \frac{\nu(f) - \nu(\partial_b f)}{b} = \epsilon(f) \right\}$$

and $b(f) = \min I(f)$.

Lemma 2.3. *Let Q be a key polynomial and take $f, g \in K[x]$ such that*

$$\deg(f) < \deg(Q)$$

and

$$\deg(g) < \deg(Q).$$

Then for $\epsilon := \epsilon(Q)$ and any $b \in \mathbb{N}$ we have the following:

(i): $\nu(\partial_b(fg)) > \nu(fg) - b\epsilon$

(ii): If $\nu_Q(fQ+g) < \nu(fQ+g)$ and $b \in I(Q)$, then $\nu(\partial_b(fQ+g)) = \nu(fQ) - b\epsilon$;

(iii): If h_1, \dots, h_s are polynomials such that $\deg(h_i) < \deg(Q)$ for every $i = 1, \dots, s$ and $\prod_{i=1}^s h_i = qQ + r$ with $\deg(r) < \deg(Q)$, then

$$\nu(r) = \nu\left(\prod_{i=1}^s h_i\right) < \nu(qQ).$$

Proof. (i) Since $\deg(f) < \deg(Q)$ and $\deg(g) < \deg(Q)$, for each $j \in \mathbb{N}$, we have

$$\nu(\partial_j f) > \nu(f) - j\epsilon \text{ and } \nu(\partial_j g) > \nu(g) - j\epsilon.$$

This, and the fact that

$$\partial_b(fg) = \sum_{j=0}^b \partial_j f \partial_{b-j} g,$$

imply that

$$\nu(\partial_b(fg)) \geq \min_{0 \leq j \leq b} \{\nu(\partial_j f) + \nu(\partial_{b-j} g)\} > \nu(fg) - b\epsilon.$$

(ii) If $\nu_Q(fQ+g) < \nu(fQ+g)$, then $\nu(fQ) = \nu(g)$. Hence,

$$\nu(\partial_b g) > \nu(g) - b\epsilon = \nu(fQ) - b\epsilon.$$

Moreover, for every $j \in \mathbb{N}$, we have

$$\nu(\partial_j f \partial_{b-j} Q) = \nu(\partial_j f) + \nu(\partial_{b-j} Q) > \nu(f) - j\epsilon + \nu(Q) - (b-j)\epsilon = \nu(fQ) - b\epsilon.$$

Therefore,

$$\nu(\partial_b(fQ+g)) = \nu\left(f\partial_b Q + \sum_{j=1}^b \partial_j f \partial_{b-j} Q + \partial_b g\right) = \nu(fQ) - b\epsilon.$$

(iii) We proceed by induction on s . If $s = 1$, then $h_1 = qQ + r$ with

$$\deg(h_1) < \deg(Q),$$

which implies that $h_1 = r$ and $q = 0$. Our result follows immediately.

Next, consider the case $s = 2$. Take $f, g \in K[x]$ such that $\deg(f) < \deg(Q)$, $\deg(g) < \deg(Q)$ and write $fg = qQ + r$ with $\deg(r) < \deg(Q)$. Then

$$\deg(q) < \deg(Q)$$

and for $b \in I(Q)$ we have

$$\nu(\partial_b(qQ)) = \nu\left(\sum_{j=0}^b \partial_j q \partial_{b-j} Q\right) = \nu(qQ) - b\epsilon.$$

This and part (i) imply that

$$\begin{aligned} \nu(qQ) - b\epsilon &= \nu(\partial_b(qQ)) = \nu(\partial_b(fg) - \partial_b(r)) \\ &\geq \min\{\nu(\partial_b(fg)), \nu(\partial_b(r))\} \\ &> \min\{\nu(fg), \nu(r)\} - b\epsilon. \end{aligned}$$

and consequently

$$(6) \quad \nu(r) = \nu(fg) < \nu(qQ).$$

Assume now that $s > 2$ and define $h := \prod_{i=1}^{s-1} h_i$. Write $h = q_1Q + r_1$ with $\deg(r_1) < \deg(Q)$. Then by the induction hypothesis we have

$$\nu(r_1) = \nu(h) < \nu(q_1Q)$$

and hence

$$\nu\left(\prod_{i=1}^s h_i\right) = \nu(r_1 h_s) < \nu(q_1 h_s Q).$$

Write $r_1 h_s = q_2Q + r_2$. Then, by equation (6) we have

$$\nu(r_2) = \nu(r_1 h_s) < \nu(q_2Q).$$

If $\prod_{i=1}^s h_i = qQ + r$ with $\deg(r) < \deg(Q)$, then

$$qQ + r = \prod_{i=1}^s h_i = h h_s = (q_1Q + r_1) h_s = q_1 h_s Q + r_1 h_s = q_1 h_s Q + q_2Q + r_2$$

and hence $q = q_1 h_s + q_2$ and $r = r_2$. Therefore,

$$\nu(qQ) \geq \min\{\nu(q_1 h_s Q), \nu(q_2 Q)\} > \nu(r_1 h_s) = \nu(r) = \nu\left(\prod_{i=1}^s h_i\right).$$

This is what we wanted to prove. \square

We denote by p the **exponent characteristic** of K , that is, $p = 1$ if $\text{char}(K) = 0$ and $p = \text{char}(K)$ if $\text{char}(K) > 0$.

Proposition 2.4. *Let $Q \in K[x]$ be a key polynomial and set $\epsilon := \epsilon(Q)$. Then the following hold:*

- (i): *Every element in $I(Q)$ is a power of p ;*
- (ii): *Q is irreducible.*

Proof. (i) Take $b \in I(Q)$ and assume, aiming for contradiction, that b is not a power of p . Write $b = p^t r$ where $r > 1$ is prime to p . Then, by Lemma 6 of [1], $\binom{b}{p^t}$ is prime to p and hence $\nu\left(\binom{b}{p^t}\right) = 0$. Since $\binom{b}{p^t} \partial_b = \partial_{p^t} \circ \partial_{b'}$ for $b' = b - p^t$, we have

$$\nu(\partial_{b'} Q) - \nu(\partial_b Q) = \nu(\partial_{b'} Q) - \nu(\partial_{p^t}(\partial_{b'} Q)) \leq p^t \epsilon(\partial_{b'}(Q)) < p^t \epsilon$$

because $\deg(\partial_{b'} Q) < \deg(Q)$ and Q is a key polynomial. Hence,

$$b\epsilon = \nu(Q) - \nu(\partial_b Q) = \nu(Q) - \nu(\partial_{b'} Q) + \nu(\partial_{b'} Q) - \nu(\partial_b Q) < b'\epsilon + p^t \epsilon = b\epsilon,$$

which gives the desired contradiction.

(ii) If $Q = gh$ for non-constant polynomials $g, h \in K[x]$, then by Lemma 2.3 (i), we would have for $b \in I(Q)$ that

$$\nu(\partial_b Q) > \nu(Q) - b\epsilon,$$

which is a contradiction to the definition of b and ϵ . \square

We present an example to show that ν_q does not need to be a valuation for a general polynomial $q(x) \in K[x]$.

Example 2.5. Consider a valuation ν in $K[x]$ such that $\nu(x) = \nu(a) = 1$ for some $a \in K$. Take $q(x) = x^2 + 1$ (which can be irreducible, for instance, if $K = \mathbb{R}$ or $K = \mathbb{F}_p$ and -1 is not a quadratic residue mod p). Since $x^2 - a^2 = (x^2 + 1) - (a^2 + 1)$ we have

$$\nu_q(x^2 - a^2) = \min\{\nu(x^2 + 1), \nu(a^2 + 1)\} = 0.$$

On the other hand, $\nu_q(x + a) = \nu(x + a) \geq \min\{\nu(a), \nu(x)\} = 1$ (and the same holds for $\nu_q(x - a)$). Hence

$$\nu_q(x^2 - a^2) = 0 < 1 + 1 \leq \nu_q(x - a) + \nu_q(x + a)$$

which shows that ν_q is not a valuation.

If $f = f_0 + f_1 q + \dots + f_n q^n$ is the q -standard decomposition of f we set

$$S_q(f) := \{i \in \{0, \dots, n\} \mid \nu(f_i q^i) = \nu_q(f)\} \text{ and } \delta_q(f) = \max S_q(f).$$

Proposition 2.6. *If Q is a key polynomial, then ν_Q is a valuation of $K[x]$.*

Proof. One can easily see that $\nu_Q(f + g) \geq \min\{\nu_Q(f), \nu_Q(g)\}$ for every $f, g \in K[x]$. It remains to prove that $\nu_Q(fg) = \nu_Q(f) + \nu_Q(g)$ for every $f, g \in K[x]$. Assume first that $\deg(f) < \deg(Q)$ and $\deg(g) < \deg(Q)$ and let $fg = aQ + c$ be the Q -standard expansion of fg . By Lemma 2.3 (iii) we have

$$\nu(fg) = \nu(c) < \nu(aQ)$$

and hence

$$\nu_Q(fg) = \min\{\nu(aQ), \nu(c)\} = \nu(c) = \nu(fg) = \nu(f) + \nu(g) = \nu_Q(f) + \nu_Q(g).$$

Now assume that $f, g \in K[x]$ are any polynomials and consider the Q -expansions

$$f = f_0 + \dots + f_n Q^n \text{ and } g = g_0 + \dots + g_m Q^m$$

of f and g . Then, using the first part of the proof, we obtain

$$\nu_Q(fg) \geq \min_{i,j} \{\nu_Q(f_i g_j Q^{i+j})\} = \min_{i,j} \{\nu_Q(f_i Q^i) + \nu_Q(g_j Q^j)\} = \nu_Q(f) + \nu_Q(g).$$

For each $i \in \{0, \dots, n\}$ and $j \in \{0, \dots, m\}$, let $f_i g_j = a_{ij} Q + c_{ij}$ be the Q -standard expansion of $f_i g_j$. Then, by Lemma 2.3 (iii), we have

$$\nu(f_i Q^i) + \nu(g_j Q^j) = \nu(f_i g_j) + \nu(Q^{i+j}) = \nu(c_{ij}) + \nu(Q^{i+j}) = \nu(c_{ij} Q^{i+j}).$$

Let

$$i_0 = \min\{i \mid \nu_Q(f) = \nu(f_i Q^i)\} \text{ and } j_0 = \min\{j \mid \nu_Q(g) = \nu(g_j Q^j)\},$$

and set $k_0 := i_0 + j_0$. Then for every $i < i_0$ or $j < j_0$ we have

$$(7) \quad \min\{\nu(a_{ij} Q^{i+j+1}), \nu(c_{ij} Q^{i+j})\} = \nu(f_i Q^i) + \nu(g_j Q^j) > \nu(c_{i_0 j_0} Q^{k_0}).$$

Let $fg = a_0 + a_1 Q + \dots + a_r Q^r$ be the Q -standard expansion of fg . Then

$$a_{k_0} = \sum_{i+j+1=k_0} a_{ij} + \sum_{i+j=k_0} c_{ij}.$$

This and equation (7) give us that

$$\nu(a_{k_0} Q^{k_0}) = \nu(c_{i_0 j_0} Q^{k_0}) = \nu(f_{i_0} Q^{i_0}) + \nu(g_{j_0} Q^{j_0}) = \nu_Q(f) + \nu_Q(g).$$

Therefore,

$$\nu_Q(fg) = \min_{0 \leq k \leq r} \{\nu(a_k Q^k)\} \leq \nu_Q(f) + \nu_Q(g),$$

which completes the proof. \square

Proposition 2.7. *Let $Q \in K[x]$ be a key polynomial and set $\epsilon := \epsilon(Q)$. For any $f \in K[x]$ the following hold:*

(i): *For any $b \in \mathbb{N}$ we have*

$$(8) \quad \frac{\nu_Q(f) - \nu_Q(\partial_b f)}{b} \leq \epsilon;$$

(ii): *If $S_Q(f) \neq \{0\}$, then the equality in (8) holds for some $b \in \mathbb{N}$;*

(iii): *If for some $b \in \mathbb{N}$, the equality in (8) holds and $\nu_Q(\partial_b f) = \nu(\partial_b f)$, then $\epsilon(f) \geq \epsilon$. If in addition, $\nu(f) > \nu_Q(f)$, then $\epsilon(f) > \epsilon$.*

Fix a key polynomial Q and $h \in K[x]$ with $\deg(h) < \deg(Q)$. Then, for every $b \in \mathbb{N}$ the Leibnitz rule for derivation gives us that

$$(9) \quad \partial_b(hQ^n) = \sum_{b_0 + \dots + b_r = b} T_b(b_0, \dots, b_r)$$

where

$$T_b(b_0, \dots, b_r) := \partial_{b_0} h \left(\prod_{i=1}^r \partial_{b_i} Q \right) Q^{n-r}.$$

In order to prove Proposition 2.7, we will need the following result:

Lemma 2.8. *Let Q be a key polynomial, $h \in K[x]$ with $\deg(h) < \deg(Q)$ and set $\epsilon := \epsilon(Q)$. For any $b \in \mathbb{N}$ we have*

$$\nu_Q(T_b(b_0, \dots, b_r)) \geq \nu(hQ^n) - b\epsilon.$$

Moreover, if either $b_0 > 0$ or $b_i \notin I(Q)$ for some $i = 1, \dots, r$, then

$$\nu_Q(T_b(b_0, \dots, b_r)) > \nu(hQ^n) - b\epsilon.$$

Proof. Since $\deg(h) < \deg(Q)$ and Q is a key polynomial we have $\epsilon(h) < \epsilon$. Hence, if $b_0 > 0$ we have

$$\nu(\partial_{b_0} h) \geq \nu(h) - b_0\epsilon(h) > \nu(h) - b_0\epsilon.$$

On the other hand, for every $i = 1, \dots, r$, by definition of ϵ we have

$$\nu(\partial_{b_i} Q) \geq \nu(Q) - b_i\epsilon,$$

and if $b_i \notin I(Q)$ we have

$$\nu(\partial_{b_i} Q) > \nu(Q) - b_i\epsilon.$$

Since $\nu_Q(\partial_{b_0} h) = \nu(\partial_{b_0} h)$ and $\nu_Q(\partial_{b_i} Q) = \nu(\partial_{b_i} Q)$, we have

$$\begin{aligned} \nu_Q(T_b(b_0, \dots, b_r)) &= \nu_Q\left(\partial_{b_0} h \left(\prod_{i=1}^r \partial_{b_i} Q\right) Q^{n-r}\right) \\ &= \nu_Q(\partial_{b_0} h) + \sum_{i=1}^r \nu_Q(\partial_{b_i} Q) + (n-r)\nu_Q(Q) \\ &\geq \nu(h) - b_0\epsilon + \sum_{i=1}^r (\nu(Q) - b_i\epsilon) + (n-r)\nu(Q) \\ &\geq \nu(hQ^n) - b\epsilon. \end{aligned}$$

Moreover, if $b_0 > 0$ or $b_i \notin I(Q)$ for some $i = 1, \dots, r$, then the inequality above is strict. \square

Corollary 2.9. *For every $b \in \mathbb{N}$ we have $\nu_Q(\partial_b(aQ^n)) \geq \nu(aQ^n) - b\epsilon$.*

Proof of Proposition 2.7. (i) Take any $f \in K[x]$ and consider its Q -standard expansion $f = f_0 + f_1Q + \dots + f_nQ^n$. For each $i = 0, \dots, n$, Corollary 2.9 gives us that

$$\nu_Q(\partial_b(f_iQ^i)) \geq \nu(f_iQ^i) - b\epsilon.$$

Hence,

$$\nu_Q(\partial_b(f)) \geq \min_{0 \leq i \leq n} \{\nu_Q(f_iQ^i)\} \geq \min_{0 \leq i \leq n} \{\nu(f_iQ^i) - b\epsilon\} = \nu_Q(f) - b\epsilon.$$

(ii) Assume that $S_Q(f) \neq \{0\}$ and set $j_0 = \min S_Q(f)$. Then $j_0 = p^e r$ for some $e \in \mathbb{N} \cup \{0\}$ and some $r \in \mathbb{N}$ with $(r, p) = 1$. We set $b := p^e b(Q)$ and will prove that $\nu_Q(\partial_b(f)) = \nu_Q(f) - b\epsilon$.

Write

$$f_{j_0}(\partial_{b(Q)} Q)^{p^e} = rQ + h$$

for some $r, h \in K[x]$ and $\deg(h) < \deg(Q)$ (note that $h \neq 0$ because Q is irreducible and $Q \nmid f_{j_0}$ and $Q \nmid \partial_{b(Q)}Q$). Then Lemma 2.3 (iii) gives us that

$$\nu(h) = \nu\left(f_{j_0}(\partial_{b(Q)}Q)^{p^e}\right).$$

This implies that

$$(10) \quad \nu\left(hQ^{j_0-p^e}\right) = \nu_Q(f) - b\epsilon.$$

Indeed, we have

$$\begin{aligned} \nu\left(hQ^{j_0-p^e}\right) &= \nu(h) + \nu\left(Q^{j_0-p^e}\right) = \nu\left(f_{j_0}(\partial_{b(Q)}Q)^{p^e}\right) + \nu\left(Q^{j_0-p^e}\right) \\ &= \nu(f_{j_0}) + p^e\nu\left(\partial_{b(Q)}Q\right) + (j_0 - p^e)\nu(Q) \\ &= \nu(f_{j_0}) + p^e(\nu(Q) - b(Q)\epsilon) + (j_0 - p^e)\nu(Q) \\ &= \nu(f_{j_0}) + j_0\nu(Q) - p^eb(Q)\epsilon \\ &= \nu(f_{j_0}Q^{j_0}) - p^eb(Q)\epsilon = \nu_Q(f) - b\epsilon. \end{aligned}$$

Since $f = f_0 + f_1Q + \dots + f_nQ^n$, we have $\partial_b(f) = \partial_b(f_0) + \partial_b(f_1Q) + \dots + \partial_b(f_nQ^n)$. For each $j = 0, \dots, n$, if $j \notin S_Q(f)$, then

$$\nu_Q\left(\partial_b(f_jQ^j)\right) \geq \nu_Q(f_jQ^j) - b\epsilon > \nu_Q(f) - b\epsilon.$$

We set

$$h_1 = \sum_{j \notin S_Q(f)} f_jQ^j.$$

Then $\nu_Q(h_1) > \nu_Q(f) - b\epsilon$.

For each $j \in S_Q(f)$ the term $\partial_b(f_jQ^j)$ can be written as a sum of terms of the form $T_b(b_0, \dots, b_r)$. For each $T_b(b_0, \dots, b_r)$ we have the following cases:

Case 1: $b_0 > 0$ or $b_i \notin I(Q)$ for some i .

In this case, by Lemma 2.8 we have $\nu_Q(T_b(b_0, \dots, b_r)) > \nu_Q(f) - b\epsilon$. In particular, if h_2 is the sum of all these terms, then $\nu_Q(h_2) > \nu_Q(f) - b\epsilon$.

Case 2: $b_0 = 0$ and $b_i \in I(Q)$ for every $i = 1, \dots, r$ but $b_{i_0} \neq b(Q)$ for some $i_0 = 1, \dots, r$.

This implies, in particular, that $j \geq j_0$ and since $b = p^eb(Q)$ we must have $r < p^e$. Hence

$$T_b(b_0, b_1, \dots, b_r) = \partial_{b_0}f_j \left(\prod_{i=1}^r \partial_{b_i}Q \right) Q^{j-r} = sQ^{j_0-p^e+1}$$

for some $s \in K[x]$.

Case 3: $b_0 = 0$, $j > j_0$ and $b_i = b(Q)$ for every $i = 1, \dots, r$.

Since $b = p^eb(Q)$, $b_i = b(Q)$ and $\sum_{i=1}^r b_i = b$ we must have $r = p^e$. Hence

$$T_b(b_0, b_1, \dots, b_r) = f_j \left(\partial_{b(Q)}Q \right)^{p^e} Q^{j-p^e} = s'Q^{j_0-p^e+1}$$

for some $s' \in K[x]$.

Case 4: $b_0 = 0$, $j = j_0$ and $b_i = b(Q)$ for every $i = 1, \dots, r$.

In this case we have

$$\begin{aligned}
 T_b(b_0, b_1, \dots, b_r) &= f_{j_0} (\partial_{b(Q)} Q)^{p^e} Q^{j_0 - p^e} \\
 (11) \qquad \qquad \qquad &= (h - rQ) Q^{j_0 - p^e} \\
 &= hQ^{j_0 - p^e} - rQ^{j_0 - p^e + 1}.
 \end{aligned}$$

Observe that the number of times that the term (11) appears in $\partial_b(f_{j_0} Q^{j_0})$ is $\binom{j_0}{p^e}$, that is, the number of ways that one can choose a subset with p^e elements in a set of j_0 elements.

Therefore, we can write

$$\partial_b(f) = \binom{j_0}{p^e} hQ^{j_0 - p^e} + \left(s + s' - \binom{j_0}{p^e} r \right) Q^{j_0 - p^e + 1} + h_1 + h_2$$

Since $p \nmid \binom{j_0}{p^e}$ the equation (10) gives us that

$$\nu \left(\binom{j_0}{p^e} hQ^{j_0 - p^e} \right) = \nu_Q(f) - b\epsilon.$$

Then

$$\nu_Q \left(\binom{j_0}{p^e} hQ^{j_0 - p^e} + \left(s + s' - \binom{j_0}{p^e} r \right) Q^{j_0 - p^e + 1} \right) \leq \nu_Q(f) - b\epsilon.$$

This and the fact that $\nu_Q(h_1 + h_2) > \nu_Q(f) - b\epsilon$ imply that $\nu_Q(\partial_b(f)) \leq \nu_Q(f) - b\epsilon$.

This concludes the proof of **(ii)**.

(iii) The assumptions on b give us

$$\frac{\nu_Q(f) - \nu_Q(\partial_b f)}{b} = \epsilon$$

and

$$\nu_Q(\partial_b f) = \nu(\partial_b f).$$

Consequently,

$$\epsilon(f) \geq \frac{\nu(f) - \nu(\partial_b f)}{b} \geq \frac{\nu_Q(f) - \nu_Q(\partial_b f)}{b} = \epsilon.$$

In the inequality above, one can see that if $\nu(f) > \nu_Q(f)$, then $\epsilon(f) > \epsilon$.

□

Proposition 2.10. *For two key polynomials $Q, Q' \in K[x]$ we have the following:*

- (i):** *If $\deg(Q) < \deg(Q')$, then $\epsilon(Q) < \epsilon(Q')$;*
- (ii):** *If $\epsilon(Q) < \epsilon(Q')$, then $\nu_Q(Q') < \nu(Q')$;*
- (iii):** *If $\deg(Q) = \deg(Q')$, then*

$$(12) \qquad \nu(Q) < \nu(Q') \iff \nu_Q(Q') < \nu(Q') \iff \epsilon(Q) < \epsilon(Q').$$

Proof. Item **(i)** follows immediately from the the definition of key polynomial (in fact, the same holds if we substitute Q for any $f \in K[x]$).

In order to prove **(ii)** we set $\epsilon := \epsilon(Q)$ and $b' := b(Q')$. By **(i)** of Proposition 2.7, we have

$$\nu_Q(Q') \leq \nu_Q(\partial_{b'} Q') + b'\epsilon.$$

Since $\epsilon(Q) < \epsilon(Q')$, we also have

$$\nu(\partial_{b'}Q') + b'\epsilon < \nu(\partial_{b'}Q') + b'\epsilon(Q') = \nu(Q').$$

This, and the fact that $\nu_Q(\partial_{b'}Q') \leq \nu(\partial_{b'}Q')$, imply that $\nu_Q(Q') < \nu(Q')$.

Now assume that $\deg(Q) = \deg(Q')$ and let us prove (12). Since

$$\deg(Q) = \deg(Q')$$

and both Q and Q' are monic, the Q -standard expansion of Q' is given by

$$Q' = Q + (Q - Q').$$

Hence

$$\nu_Q(Q') = \min\{\nu(Q), \nu(Q - Q')\}.$$

The first equivalence follows immediately from this. In view of part (ii), it remains to prove that if $\nu_Q(Q') < \nu(Q')$, then $\epsilon(Q) < \epsilon(Q')$. Since $\nu_Q(Q') < \nu(Q')$ we have $S_Q(Q') \neq \{0\}$. Hence, by Proposition 2.7 (ii), the equality holds in (8) (for $f = Q'$) for some $b \in \mathbb{N}$. Moreover, since $\deg(Q) = \deg(Q')$, we have $\deg(\partial_b Q') < \deg(Q)$ and consequently $\nu_Q(\partial_b Q') = \nu(\partial_b Q')$. Then Proposition 2.7 (iii) implies that $\epsilon(Q) < \epsilon(Q')$. \square

For a key polynomial $Q \in K[x]$, let

$$\alpha(Q) := \min\{\deg(f) \mid \nu_Q(f) < \nu(f)\}$$

(if $\nu_Q = \nu$, then set $\alpha(Q) = \infty$) and

$$\Psi(Q) := \{f \in K[x] \mid f \text{ is monic, } \nu_Q(f) < \nu(f) \text{ and } \alpha(Q) = \deg(f)\}.$$

Lemma 2.11. *If Q is a key polynomial, then every element $Q' \in \Psi(Q)$ is also a key polynomial. Moreover, $\epsilon(Q) < \epsilon(Q')$.*

Proof. By assumption, we have $\nu_Q(Q') < \nu(Q')$, hence $S_Q(Q') \neq \{0\}$. This implies, by Proposition 2.7 (ii), that there exists $b \in \mathbb{N}$ such that

$$\nu_Q(Q') - \nu_Q(\partial_b Q') = b\epsilon(Q).$$

Since $\deg(\partial_b Q') < \deg(Q') = \alpha(Q)$, we have $\nu_Q(\partial_b Q') = \nu(\partial_b Q')$. Consequently, by Proposition 2.7 (iii), $\epsilon(Q) < \epsilon(Q')$.

Now take any polynomial $f \in K[x]$ such that $\deg(f) < \deg(Q') = \alpha(Q)$. In particular, $\nu_Q(f) = \nu(f)$. Moreover, for every $b \in \mathbb{N}$, $\deg(\partial_b f) < \deg(Q') = \alpha(Q)$ which implies that $\nu_Q(\partial_b f) = \nu(\partial_b f)$. Then, for every $b \in \mathbb{N}$,

$$\frac{\nu(f) - \nu(\partial_b f)}{b} = \frac{\nu_Q(f) - \nu_Q(\partial_b f)}{b} \leq \epsilon(Q) < \epsilon(Q').$$

This implies that $\epsilon(f) < \epsilon(Q')$, which shows that Q' is a key polynomial. \square

Theorem 2.12. *A polynomial Q is a key polynomial if and only if there exists a key polynomial $Q_- \in K[x]$ such that $Q \in \Psi(Q_-)$ or the following conditions hold:*

$$(\mathbf{K1}): \alpha(Q_-) = \deg(Q_-)$$

(K2): the set $\{\nu(Q') \mid Q' \in \Psi(Q_-)\}$ does not contain a maximal element

(K3): $\nu_{Q'}(Q) < \nu(Q)$ for every $Q' \in \Psi(Q_-)$

(K4): Q has the smallest degree among polynomials satisfying **(K3)**.

Proof. We will prove first that if such Q_- exists, then Q is a key polynomial. The case when $Q \in \Psi(Q_-)$ follows from Lemma 2.11. Assume now that **(K1)** - **(K4)** hold. Take $f \in K[x]$ such that $\deg(f) < \deg(Q)$. This implies that $\deg(\partial_b Q) < \deg(Q)$ and $\deg(\partial_b f) < \deg(Q)$ for every $b \in \mathbb{N}$. Hence, by **(K4)**, there exists $Q' \in \Psi(Q_-)$ such that

$$\nu_{Q'}(f) = \nu(f), \nu_{Q'}(\partial_b f) = \nu(\partial_b f) \text{ and } \nu_{Q'}(\partial_b Q) = \nu(\partial_b Q) \text{ for every } b \in \mathbb{N}.$$

We claim that $\epsilon(Q') < \epsilon(Q)$. If not, by Proposition 2.10 **(i)**, we would have $\deg(Q) \leq \deg(Q')$. Since $\nu_{Q'}(Q) < \nu(Q)$, this implies that $\deg(Q) = \deg(Q')$. This and Proposition 2.10 **(iii)** give us that $\epsilon(Q') < \epsilon(Q)$ which is a contradiction.

Now,

$$\epsilon(f) \leq \frac{\nu(f) - \nu(\partial_b f)}{b} = \frac{\nu_{Q'}(f) - \nu_{Q'}(\partial_b f)}{b} \leq \epsilon(Q') < \epsilon(Q).$$

Hence Q is a key polynomial.

For the converse, take a key polynomial $Q \in K[x]$ and consider the set

$$\mathcal{S} := \{Q' \in K[x] \mid Q' \text{ is a key polynomial and } \nu_{Q'}(Q) < \nu(Q)\}.$$

Observe that $\mathcal{S} \neq \emptyset$. Indeed, if $\deg(Q) > 1$, then every key polynomial $x - a \in \mathcal{S}$. If $Q = x - a$, then there exists $b \in K$ such that $\nu(b) < \min\{\nu(a), \nu(x)\}$. Therefore, $x - b \in \mathcal{S}$.

If there exists a key polynomial $Q_- \in \mathcal{S}$ such that $\deg(Q) = \deg(Q_-)$, then we have $Q \in \Psi(Q_-)$ and we are done. Hence, assume that every polynomial $Q' \in \mathcal{S}$ has degree smaller than $\deg(Q)$.

Assume that there exists $Q_- \in \mathcal{S}$ such that for every $Q' \in \mathcal{S}$ we have

$$(13) \quad (\deg(Q_-), \nu(Q_-)) \geq ((\deg(Q'), \nu(Q')))$$

in the lexicographical ordering. We claim that $Q \in \Psi(Q_-)$. If not, there would exist a key polynomial Q'' such that $\nu_{Q_-}(Q'') < \nu(Q'')$ and $\deg(Q'') < \deg(Q)$. Since $\deg(Q'') < \deg(Q)$ Proposition 2.10 **(i)** and **(ii)** give us that $\nu_{Q''}(Q) < \nu(Q)$. Hence $Q'' \in \mathcal{S}$. The inequality (13) gives us that $\deg(Q'') \leq \deg(Q_-)$. On the other hand, since $\nu_{Q_-}(Q'') < \nu(Q'')$ we must have $\deg(Q_-) = \deg(Q'')$. Hence, Proposition 2.10 **(iii)** gives us that $\nu(Q_-) < \nu(Q'')$ and this is a contradiction to the inequality (13).

Now assume that for every $Q' \in \mathcal{S}$, there exists $Q'' \in \mathcal{S}$ such that

$$(14) \quad (\deg(Q'), \nu(Q')) < (\deg(Q''), \nu(Q''))$$

in the lexicographical ordering. Take $Q_- \in \mathcal{S}$ such that $\deg(Q_-) \geq \deg(Q')$ for every $Q' \in \mathcal{S}$. We will show that the conditions **(K1)** - **(K4)** are satisfied. By (14), there exists $Q'' \in \mathcal{S}$ such that

$$(15) \quad (\deg(Q_-), \nu(Q_-)) < (\deg(Q''), \nu(Q'')).$$

In particular, $\deg(Q_-) = \deg(Q'')$ and $\nu(Q_-) < \nu(Q'')$. Proposition 2.10 (iii) gives us that $\nu_{Q_-}(Q'') < \nu(Q'')$. Hence $\alpha(Q_-) = \deg(Q_-)$ and we have proved **(K1)**. If $Q' \in \Psi(Q_-)$, then $\deg(Q') = \deg(Q_-) < \deg(Q)$ and hence $\nu_{Q'}(Q) < \nu(Q)$. This implies that $Q' \in \mathcal{S}$. The equation (15) tells us that $\{\nu(Q') \mid Q' \in \Psi(Q_-)\}$ has no maximum, so we have proved **(K2)**. Now take any element $Q' \in \Psi(Q_-)$. Then $\deg(Q') < \deg(Q)$ and Proposition 2.10 (i) and (ii) give us that $\nu_{Q'}(Q) < \nu(Q)$. This proves **(K3)**. Take a polynomial \tilde{Q} with $\nu_{Q'}(\tilde{Q}) < \nu(\tilde{Q})$ for every $Q' \in \Psi(Q_-)$ with minimal degree possible. We want to prove that $\deg(\tilde{Q}) = \deg(Q)$. Assume, aiming for a contradiction, that $\deg(\tilde{Q}) < \deg(Q)$. The first part of the proof gives us that \tilde{Q} is a key polynomial. Fix $Q' \in \Psi(Q_-)$. Then $\nu_{Q'}(\tilde{Q}) < \nu(\tilde{Q})$ and consequently $\deg(\tilde{Q}) = \deg(Q') = \deg(Q_-)$. Therefore $\nu(Q') < \nu(\tilde{Q})$ for every $Q' \in \Psi(Q_-)$, which is a contradiction to (14). This concludes our proof. \square

Definition 2.13. When conditions **(K1)** - **(K4)** of Theorem 2.12 are satisfied, we say that Q is a **limit key polynomial**.

Remark 2.14. Observe that as a consequence of the proof we obtain that

$$\epsilon(Q_-) < \epsilon(Q).$$

Proof of Theorem 1.1. Consider the set

$$\Gamma_0 := \{\nu(x - a) \mid a \in K\}.$$

We have two possibilities:

- Γ_0 has a maximal element

Set $Q_0 := x - a_0$ where $a_0 \in K$ is such that $\nu(x - a_0)$ is a maximum of Γ_0 . If $\nu = \nu_{Q_0}$ we are done, so assume that $\nu \neq \nu_{Q_0}$. If the set

$$\{\nu(Q) \mid Q \in \Psi(Q_0)\}$$

has a maximum, choose $Q_1 \in \Psi(Q_0)$ such that $\nu(Q_1)$ is this maximum. If not, choose Q_1 as any polynomial in $\Psi(Q_0)$. Set $\Lambda_1 := \{Q_0, Q_1\}$ (ordered by $Q_0 < Q_1$).

- Γ_0 does not have a maximal element

For every $\gamma \in \Gamma_0$ set $Q_\gamma := x - a_\gamma$ for some $a_\gamma \in K$ such that $\nu(x - a_\gamma) = \gamma$. If for every $f \in K[x]$, there exists $\gamma \in \Gamma_0$ such that $\nu(f) = \nu_{Q_\gamma}(f)$ we are done. If not, let Q be a polynomial of minimal degree among all the polynomials for which $\nu_{Q_\gamma}(Q) < \nu(Q)$ for every $\gamma \in \Gamma_0$. If $\alpha(Q) = \deg(Q)$ and the set $\{\nu(Q') \mid Q' \in \Psi(Q)\}$ contains a maximal element, choose $Q_1 \in \Psi(Q)$ such that $\nu(Q_1) \geq \nu(Q')$ for every $Q' \in \Psi(Q)$. If not, set $Q_1 := Q$. Set $\Lambda_1 := \{Q_\gamma \mid \gamma \in \Gamma_0\} \cup \{Q_1\}$ (ordered by $Q_1 > Q_\gamma$ for every $\gamma \in \Gamma$ and $Q_\gamma > Q_{\gamma'}$ if $\gamma > \gamma'$).

Observe that in either case, $\deg(Q_1) > \deg(Q_0)$ and for $Q, Q' \in \Lambda_1$, $Q < Q'$ if and only if $\epsilon(Q) < \epsilon(Q')$. Moreover, if $\alpha(Q_1) = \deg(Q_1)$, then $\{\nu(Q) \mid Q \in \Psi(Q_1)\}$ does not have a maximum.

Assume that for some $i \in \mathbb{N}$, there exists a totally ordered set Λ_i consisting of key polynomials with the following properties:

- (i): there exist $Q_0, Q_1, \dots, Q_i \in \Lambda_i$ such that Q_i is the last element of Λ_i and $\deg(Q_0) < \deg(Q_1) < \dots < \deg(Q_i)$.
- (ii): if $\alpha(Q_i) = \deg(Q_i)$, then $\Gamma_i := \{\nu(Q) \mid Q \in \Psi(Q_i)\}$ does not have a maximum.
- (iii): for $Q, Q' \in \Lambda_i$, $Q < Q'$ if and only if $\epsilon(Q) < \epsilon(Q')$.

If $\nu_{Q_i} \neq \nu$, then we will construct a set Λ_{i+1} of key polynomials having the same properties (changing i by $i+1$).

Since $\nu_{Q_i} \neq \nu$, the set $\Psi(Q_i)$ is not empty. We have two cases:

- $\alpha(Q_i) > \deg(Q_i)$.

If Γ_i has a maximum, take $Q_{i+1} \in \Psi(Q_i)$ such that $\nu(Q_{i+1}) \geq \Gamma_i$. Otherwise, choose Q_{i+1} to be any element of $\Psi(Q_i)$. Observe that if $\alpha(Q_{i+1}) = \deg(Q_{i+1})$, then Γ_{i+1} does not have a maximum. Set $\Lambda_{i+1} = \Lambda_i \cup \{Q_{i+1}\}$ with the extension of the order in Λ_i obtained by setting $Q_{i+1} > Q$ for every $Q \in \Lambda_i$.

- $\alpha(Q_i) = \deg(Q_i)$.

By assumption, the set Γ_i does not have a maximum. For each $\gamma \in \Gamma_i$, choose a polynomial $Q_\gamma \in \Psi(Q_i)$ such that $\nu(Q_\gamma) = \gamma$. If for every $f \in K[x]$, there exists $\gamma \in \Gamma_i$ such that $\nu_{Q_\gamma}(f) = \nu(f)$, then we are done. Otherwise, choose a monic polynomial Q , of smallest degree possible, such that $\nu_{Q'}(Q) < \nu(Q)$ for every $Q' \in \Psi(Q_i)$. If $\alpha(Q) = \deg(Q)$ and $\{\nu(Q') \mid Q' \in \Psi(Q)\}$ has a maximum, we choose Q_{i+1} such that $\nu(Q_{i+1}) \geq \{\nu(Q') \mid Q' \in \Psi(Q)\}$. Otherwise we set $Q_{i+1} = Q$. Then set

$$\Lambda_{i+1} := \Lambda_i \cup \{Q_\gamma \mid \gamma \in \Gamma_i\} \cup \{Q_{i+1}\},$$

with the extension of the order of Λ_i given by

$$Q_{i+1} > Q' \text{ for every } Q' \in \Lambda_{i+1} \setminus \{Q_{i+1}\},$$

$Q_\gamma > Q'$ for every $\gamma \in \Gamma_i$ and $Q' \in \Lambda_i$ and $Q_\gamma > Q_{\gamma'}$ for $\gamma, \gamma' \in \Gamma_i$ with $\gamma > \gamma'$.

In all cases, the set Λ_{i+1} has the properties (i), (ii) and (iii).

Assume now that for every $i \in \mathbb{N}$ the sets Λ_i and Λ_{i+1} can be constructed. Then we can construct a set

$$\Lambda_\infty := \bigcup_{i=1}^{\infty} \Lambda_i$$

of key polynomials having the property that for $Q, Q' \in \Lambda_\infty$, $Q < Q'$ if and only if $\epsilon(Q) < \epsilon(Q')$ and there are polynomials $Q_0, \dots, Q_i, \dots \in \Lambda_\infty$ such that

$$\deg(Q_{i+1}) > \deg(Q_i)$$

for every $i \in \mathbb{N}$. This means that for every $f \in K[x]$ there exists $i \in \mathbb{N}$ such that $\deg(f) < \deg(Q_i)$, which implies that $\nu_{Q_i}(f) = \nu(f)$. Therefore, Λ_∞ is a complete set of key polynomials for ν . \square

Observe that at each stage, the same construction would work if we replaced Γ_i by any cofinal set Γ'_i of Γ_i . Hence, if the rank of ν is equal to 1, then we can choose

Γ'_i to have order type at most ω . Then, from the construction of the sets Λ_i and Λ_∞ , we can conclude the following:

Corollary 2.15. *If the rank of ν is equal to one, then there exists a complete sequence of key polynomials of ν with order type at most $\omega \times \omega$.*

3. PSEUDO-CONVERGENT SEQUENCES

The next two theorems justify the definitions of algebraic and transcendental pseudo-convergent sequences.

Theorem 3.1 (Theorem 2 of [1]). *If $\{a_\rho\}_{\rho < \lambda}$ is a pseudo-convergent sequence of transcendental type, without a limit in K , then there exists an immediate transcendental extension $K(z)$ of K defined by setting $\nu(f(z))$ to be the value $\nu(f(a_{\rho_f}))$ as in condition (2). Moreover, for every valuation μ in some extension $K(u)$ of K , if u is a pseudo-limit of $\{a_\rho\}_{\rho < \lambda}$, then there exists a value preserving K -isomorphism from $K(u)$ to $K(z)$ taking u to z .*

Theorem 3.2 (Theorem 3 of [1]). *Let $\{a_\rho\}_{\rho < \lambda}$ be a pseudo-convergent sequence of algebraic type, without a limit in K , $q(x)$ a polynomial of smallest degree for which (3) holds and z a root of $q(x)$. Then there exists an immediate algebraic extension of K to $K(z)$ defined as follows: for every polynomial $f(x) \in K[x]$, with $\deg f < \deg q$ we set $\nu(f(z))$ to be the value $\nu(f(a_{\rho_f}))$ as in condition (2). Moreover, if u is a root of $q(x)$ and μ is some extension $K(u)$ of K making u a pseudo-limit of $\{a_\rho\}_{\rho < \lambda}$, then there exists a value preserving K -isomorphism from $K(u)$ to $K(z)$ taking u to z .*

For the rest of this paper, let $\{a_\rho\}_{\rho < \lambda}$ be a pseudo-convergent sequence for the valued field (K, ν) , without a limit in K . For each $\rho < \lambda$, we denote $\nu_\rho = \nu_{x-a_\rho}$. For a polynomial $f(x) \in K[x]$ and $a \in K$ we consider the Taylor expansion of f at a given by

$$f(x) = f(a) + \partial_1 f(a)(x-a) + \dots + \partial_n f(a)(x-a)^n.$$

Assume that $\{a_\rho\}_{\rho < \lambda}$ fixes the value of the polynomials $\partial_i f(x)$ for every $1 \leq i \leq n$. We denote by β_i this fixed value.

Lemma 3.3 (Lemma 8 of [1]). *There is an integer h , which is a power of p , such that for sufficiently large ρ*

$$\beta_i + i\gamma_\rho > \beta_h + h\gamma_\rho \text{ whenever } i \neq h \text{ and } \nu(f(a_\rho)) = \beta_h + h\gamma_\rho.$$

Corollary 3.4. *If $\{a_\rho\}_{\rho < \lambda}$ fixes the value of $f(x)$, then $\nu_\rho(f(x)) = \nu(f(x))$. On the other hand, if $\{a_\rho\}_{\rho < \lambda}$ does not fix the value of $f(x)$, then $\nu_\rho(f(x)) < \nu(f(x))$ for every $\rho < \lambda$.*

Proof. By definition of ν_ρ we have

$$\nu_\rho(f(x)) = \min_{0 \leq i \leq n} \{\nu(\partial_i f(a_\rho)(x-a_\rho)^i)\} = \min_{0 \leq i \leq n} \{\beta_i + i\gamma_\rho\},$$

where $\beta_0 := \nu(f(a_\rho))$. This implies, using the lemma above, that

$$\nu_\rho(f(x)) = \nu(f(a_\rho)).$$

If $\{a_\rho\}_{\rho < \lambda}$ fixes the value of $f(x)$, then $\nu(f(a_\rho)) = \nu(f(x))$ for ρ sufficiently large. Thus $\nu_\rho(f(x)) = \nu(f(x))$. On the other hand, if $\{a_\rho\}_{\rho < \lambda}$ does not fix the value of $f(x)$, then $\nu(f(x)) > \nu(f(a_\rho)) = \nu_\rho(f(x))$ for every $\rho < \lambda$. \square

Proof of Theorem 1.2. If $\{a_\rho\}_{\rho < \lambda}$ is of transcendental type it fixes, for any polynomial $f(x) \in K[x]$, the values of the polynomials $\partial_i f(x)$ for every $0 \leq i \leq n$ (here $\partial_0 f := f$). Hence, Corollary 3.4 implies that $\nu_\rho(f(x)) = \nu(f(x))$ for sufficiently large $\rho < \lambda$, which is what we wanted to prove.

Now assume that $\{a_\rho\}_{\rho < \lambda}$ is of algebraic type. Take $\rho < \lambda$ such that

$$\nu(q(a_\tau)) > \nu(q(a_\sigma))$$

for every $\rho < \sigma < \tau < \lambda$ and set $Q_- = x - a_\rho$. Then

$$\nu_{Q_-}(x - a_\sigma) = \nu_{Q_-}(x - a_\rho + a_\rho - a_\sigma) = \nu(x - a_\rho) < \nu(x - a_\sigma)$$

for every $\rho < \sigma < \lambda$. This implies that $\alpha(Q_-) = 1$ and then $\alpha(Q_-) = \deg(Q_-)$. Consequently, **(K1)** is satisfied. Moreover,

$$\Psi(Q_-) = \{x - a \mid \nu_{Q_-}(x - a) < \nu(x - a)\}.$$

In order to prove **(K2)** assume, aiming for a contradiction, that $\nu(\Psi(Q_-))$ has a maximum, let us say $\nu(x - a)$. Then, in particular, $\nu(x - a) > \nu(x - a_\sigma)$ for every $\rho < \sigma < \lambda$. This implies that $a \in K$ is a limit of $\{a_\rho\}_{\rho < \lambda}$, which is a contradiction. Condition **(K3)** and **(K4)** follow immediately from Corollary 3.4 and the fact that $\{\nu(x - a_\rho) \mid \rho < \lambda\}$ is cofinal in $\nu(\Psi(Q_-))$. \square

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